$$= \int_{a=x_0}^{x_1} f(x)\, g(x)\, d\alpha(x) + \int_{x_1}^{x_2} f(x)\, g(x)\, d\alpha(x) + \cdots$$

$$+ \int_{x_{n-1}}^{x_n = b} f(x)\, g(x)\, d\alpha(x)$$

$$\Rightarrow A = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\, g(x)\, d\alpha(x)$$

$$\text{(i.e)} \quad A = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)\, g(t)\, d\alpha(t) \longrightarrow (2)$$

[$\because x$ is dummy variable of integration]

Now, consider

$$|S(P,f,g) - A| = \left| \sum_{k=1}^{n} f(t_k) \int_{x_{k-1}}^{x_k} g(t)\, d\alpha(t) - \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(t)\, g(t)\, d\alpha(t) \right|$$

$$|S(P,f,G) - A| = \left| \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} g(t) \left\{ f(t_k) - f(t) \right\} d\alpha(t) \right|$$

$$\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |g(t)|\, |f(t_k) - f(t)|\, d\alpha(t) \longrightarrow (A)$$

Since g is bounded on $[a,b]$

Let $M_g = \operatorname{Sup} \{ |g(x)| : x \in [a,b] \} \longrightarrow (5)$

Now consider,

$$M_k(f) - m_k(f) = \operatorname{Sup} \{ f(x) - f(y) : x, y \in [x_{k-1}, x_k] \}$$

$$= \operatorname{Sup} \{ |f(x) - f(y)| : x, y \in [x_{k-1}, x_k] \}$$

$$\geq |f(x) - f(y)|, \ \forall \ x, y \in [x_{k-1}, x_k]$$

$$\Rightarrow |f(x) - f(y)| \leq M_k(f) - m_k(f), \ \forall \ x, y \in [x_{k-1}, x_k] \longrightarrow (4)$$

Using (5) & (4) in (A), we get,

$$|S(P,f,G) - A| \leq \left[\sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} M_g \{ M_k(f) - m_k(f) \}\, d\alpha(t) \right]$$

[$\because |g(x)| \leq M_g$]

$$= M_g \sum_{k=1}^{n} \{ M_k(f) - m_k(f) \} \int_{x_{k-1}}^{x_k} d\alpha(t)$$

$$= M_g \sum_{k=1}^{n} \{ M_k(f) - m_k(f) \} \left[\alpha(t) \right]_{x_{k-1}}^{x_k}$$

$$= M_g \sum_{k=1}^{n} \{M_k(f) - m_k(f)\} [\alpha(x_k) - \alpha(x_{k-1})]$$

$$= M_g \sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k$$

$$= M_g \sum_{k=1}^{n} [M_k(f) - m_k(f)] |\Delta\alpha_k| \quad [\because \alpha \text{ is } \uparrow \text{ on } [a,b] \; |\Delta\alpha_k| = \Delta\alpha_k]$$

$$= M_g \{U(P,f,\alpha) - L(P,f,\alpha)\}$$

i.e.) $|S(P,f,G) - A| \leq M_g \{U(P,f,\alpha) - L(P,f,\alpha)\} \longrightarrow (5)$

Since $f \in R(\alpha)$ on $[a,b]$, f satisfies Riemann

For any given $\epsilon > 0$, $\exists$ a partition P_ϵ of $[a,b]$

Such that $\forall P \supseteq P_\epsilon$,

$$U(P,f,\alpha) - L(P,f,\alpha) < \frac{\epsilon}{M_g}$$

using the above inequality in (5) we get,

$$|S(P,f,G) - A| < M_g \frac{\epsilon}{M_g} = \epsilon$$

$$|S(P,f,G) - A| < \epsilon$$

$\Rightarrow f \in R(G)$ and on $[a,b]$ and $\int_a^b f(x) g(x) \, d\alpha(x) = \int_a^b$

$\Rightarrow f \in R(G)$ and $A = \int_a^b f(x) \, dG(x)$

$\Rightarrow f \in R(G)$ on $[a,b]$ and $\int_a^b f(x) g(x) \, d\alpha(x) = \int_a^b f(x) \, d G(x)$

By Similar arguement we can prove that

$g \in R(F)$ on $[a,b]$ and

$$\int_a^b f(x) g(x) \, d\alpha(x) = \int_a^b g(x) \, dF(x)$$

Hence the proof.

Sufficient - Conditions for Existence of Riemann-Stieltjes Integral

Theorem: 4

If f is continuous on $[a,b]$ and if α is of b.v on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$

Proof: Given,

(i) f is continuous $\Rightarrow f$ is uniform continuous on $[a,b]$

(ii) α is of bounded variation on $[a,b]$, $\alpha(c) \le \alpha(b)$ i) $\alpha(a)=\alpha(b)$ ii) $\alpha(a) < \alpha(b)$

To prove: $f \in R(\alpha)$ on $[a,b]$

ie) To prove that f satisfies the Riemann condition on $[a,b]$ w.r.t α, $\forall\ \epsilon>0\ \exists\ P_\epsilon$ of $[a,b]$ $\ni$ $\forall\ P \supseteq P_\epsilon$

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

ie) $\sum_{k=1}^{n} [M_k(f) - m_k(f)] \Delta\alpha_k < \epsilon$

Since every function of bounded variation is expressed as a difference of two increasing functions. it is enough to prove the theorems when α is increasing on $[a,b]$.

Let α be an increasing function on $[a,b]$.

Then $\alpha(a) \le \alpha(b)$ on $[a,b]$

If (i) $\alpha(a) = \alpha(b)$, then,

α is constant on $[a,b]$ $\Delta(\alpha)=0 \Rightarrow \int f d\alpha = 0$.

$\Rightarrow f \in R(\alpha)$ on $[a,b]$

$\therefore$ (ii) Assume $\alpha(a) < \alpha(b) \Rightarrow \alpha(b)-\alpha(a) > 0$ p.t $f \in R(\alpha)$ on (a,b)

p.t $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\sum [M_k(f) - m_k(f)] \Delta\alpha_k < \epsilon$

Since f is continuous on compact interval on $[a,b]$

f is uniformly continuous on $[a,b]$

$\therefore$ For any $\dfrac{\epsilon}{(\alpha(b)-\alpha(a))} > 0$, $\exists\ \delta>0$ [depending only on ϵ) such that,

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \frac{\epsilon}{(\alpha(b)-\alpha(a))} \longrightarrow (1)$$

Let P_ϵ be a partition of $[a,b]$ with $\|P_\epsilon\| < \delta$, then

$P \supseteq P_\epsilon$ implies

$$\|P\| < \|P_\epsilon\| < \delta \Rightarrow \|P\| < \delta$$

[Since length of the largest subinterval in P is less than δ, if $x, y \in [x_{k-1}, x_k]$ of P then $|x-y| < \delta$, $\forall\ k = 1, 2 \ldots n$]

$$\text{Hence (1)} \Longrightarrow |f(x)-f(y)| < \frac{\epsilon}{(\alpha(b)-\alpha(a))} \longrightarrow (2)$$

Hence $\forall\ P \supseteq P_\epsilon$,

$$M_k(f) - m_k(f) = \sup\{f(x)-f(y) : x, y \in [x_{k-1}, x_k]\}$$

$$\leq \sup\{|f(x)-f(y)| : x, y \in [x_{k-1}, x_k]\}$$

$$\Rightarrow |f(x)-f(y)| < \frac{\epsilon}{\alpha(b)-\alpha(a)}$$

$$\leq \frac{\epsilon}{(\alpha(b)-\alpha(a))}$$

$$\forall\ P \supseteq P_\epsilon, \quad M_k(f) - m_k(f) \leq \frac{\epsilon}{(\alpha(b)-\alpha(a))}$$

$$\Rightarrow \sum_{k=1}^{n} [M_k(f) - m_k(f)]\Delta\alpha_k \leq \frac{\epsilon}{(\alpha(b)-\alpha(a))} \sum_{k=1} \Delta\alpha_k$$

$$\Rightarrow \sum_{k=1}^{n} [M_k(f) - m_k(f)]\Delta\alpha_k \leq \frac{\epsilon}{(\alpha(b)-\alpha(a))} [\alpha(b)-\alpha(a)] = \epsilon$$

$$\therefore \sum_{k=1} M_k(f)\Delta\alpha_k - \sum_{k=1} m_k(f)\Delta\alpha_k < \epsilon$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon, \quad \forall\ P \supseteq P_\epsilon$$

ie) f satisfies Riemann condition w.r.t α on $[a,b]$

$f \in R(\alpha)$ on $[a,b]$

Hence proved.

Note:

If α is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$ then $f \in R(\alpha)$ on $[a,b]$

proof: By thm 4, $\alpha \in R(f)$ on $[a,b]$

By integration by parts, $f \in R(\alpha)$ on $[a,b]$.

Theorem: 5

Necessary Back [Sufficient Condition for Existence of Riemann-Integral]

Each of the following conditions is sufficient for the Existence of Riemann - Integral then $\int_a^b f(x)\,dx$

(i) f is continuous on $[a,b]$

(ii) f is of bounded variation on $[a,b]$.

proof:

To prove that:

If (i) is true then $\int_a^b f(x)\,dx$ exists

Assume that (i) is true

ie) Assume that f is continuous on $[a,b]$

Since $\alpha(x) = x$, α is increasing on $[a,b]$

By thm 4: $f \in R(\alpha)$ on $[a,b]$

$\therefore$ Riemann Integral $\int_a^b f(x)\,dx$ exists

To prove that:

If (ii) is true then $\int_a^b f\,dx$ exists

Assume that (ii) is true.

ie) Assume that f is of b.v on $[a,b]$

Since $\alpha(x)=x$, α is continuous on $[a,b]$

$\therefore$ By the note under Thm: 4 $f \in R$ on $[a,b]$

$\therefore$ Riemann - Integral $\int_a^b f\,dx$ exists.

Scanned by TapScanner

Section 7.17

(Necessary conditions for the Existence of Riemann Stieltjes integral:)

~~Theorem:~~ Statement:

Assume that α is increasing on $[a,b]$ and let $a<c<b$

Assume further that α and f is discontinuous from the righ at $x=c$

(ie) Assume that $\exists\ \epsilon>0$ such that $\forall\ \delta>0$ there are values of x and y in the interval $[c, c+\delta)$ for which $|f(x)-f(c)| \geq \epsilon$ and $|\alpha(y)-\alpha(c)| \geq \epsilon$

Then $\int_a^b f(x)\,d\alpha(x)$ cannot exist, the integral also fails to exist if α and f are discontinuous from the left at c

Proof: Given,

(i) α increasing on $[a,b]$

(ii) $a<c<b$

(iii) α and f are discontinuous from the right at $x=c$.

To prove that: $\int_a^b f\,d\alpha$ does not exist.

(ie) TP: $f \notin R(\alpha)$ on $[a,b]$

We have $f \in R(\alpha)$ iff f satisfies Riemann condition w.r.t α on $[a,b]$

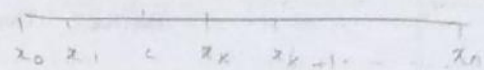
So to prove that the theorem, it suffices to prove that f does not satisfy the Riemann condition

~~Let P be a partition of $[a,b]$ such that $c \in P$~~

ETP: $U(P,f,\alpha) - L(P,f,\alpha) > \epsilon$

$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{k=1}^{n} \{ M_k(f) - m_k(f) \} \Delta\alpha_k$ $\rightarrow$ ①

Let c be the left end point of i^{th} Subinterval $[c, x_i]$ of P



then from (1),

Since each term in the sum, occurs in eqn (1) is greater than are equal to '0'.

$\therefore U(P,f,\alpha) - L(P,f,\alpha) \geq [M_i(f) - m_i(f)] \Delta\alpha_i$

$= [M_i(f) - m_i(f)] \{ \alpha(x_i) - \alpha(c) \}$ $\rightarrow$ ②

By hypothesis 'c' is a point of discontinuity of both 'f' and 'α' from the right,

$\therefore$ we can choose x_i; Such that for some $\epsilon > 0$

and $|\alpha(x_i) - \alpha(c)| \geq \epsilon$

Since α is increasing, $\alpha(x_i) - \alpha(c) \geq \epsilon$, and

$|f(x_i) - f(c)| \geq \epsilon$

Now, $M_i(f) - m_i(f) = \operatorname{Sup} \{ f(x) - f(y) : x, y \in [c, x_i] \}$

$= \operatorname{Sup} \{ |f(x) - f(y)| : x, y \in [c, x_i] \}$

$\Rightarrow M_i(f) - m_i(f) \geq |f(x_i) - f(c)| \geq \epsilon$

$\Rightarrow M_i(f) - m_i(f) \geq \epsilon$

Using the above in (2) we get

$U(P,f,\alpha) - L(P,f,\alpha) \geq \epsilon^2 > \epsilon$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) > \epsilon$

f does not Satisfy Riemann condition w.r.t α on $[a,b]$

$\Rightarrow f \notin R(\alpha)$ on $[a,b]$

$\Rightarrow \int_a^b f \, d\alpha$ does not exist

By Similar argument $\int_a^b f\,d\alpha$ cannot exist if α and f are discontinuous from left at $x=c$

Hence the proof.

Section : 7.18

Mean Value theorems for R.S.I

Theorem : 7

Q (4) First Mean Value theorem for R.S.I

Assume that α is increasing and let $f \in R(\alpha)$ on $[a,b]$. Let M and m denote respectively the Supremum and infimum of the set $\{f(x) : x \in [a,b]\}$. then there exists a real number 'c' such that $m \le c \le M$ implies

$$\int_a^b f(x)\,d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b)-\alpha(a)] \longrightarrow (1)$$

In particular, if f is continuous on $[a,b]$ then $c = f(x_0)$, for some x_0 in $[a,b]$.

Proof:

Given, (i) α increasing on $[a,b]$

(ii) $f \in R(\alpha)$ on $[a,b]$

(iii) $m = \inf\{f(x) : x \in [a,b]\}$

and $M = \sup\{f(x) : x \in [a,b]\}$

Since α increasing on $[a,b]$ we have $\alpha(a) \le \alpha(b)$

If $\alpha(a) = \alpha(b)$, then α is a constant on $[a,b]$ and $\int_a^b f\,d\alpha = 0$. Also $\alpha(b)-\alpha(a) = 0$

Thus (1) is trivially true, both sides being 0. So, assume that $\alpha(a) < \alpha(b)$.

Scanned by TapScanner

For every partition P of $[a,b]$ the upper and lower sum satisfies,

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

For, $m = g.l.b \{f(x) : x \in [a,b]\}$

$M = l.u.b \{f(x) : x \in [a,b]\}$

$m_k = g.l.b \{f(x) : x \in [x_{k-1}, x_k]\}$

$M_k = l.u.b \{f(x) : x \in [x_{k-1}, x_k]\}$

Since $[x_{k-1}, x_k] \subseteq [a,b]$, then

$$m \le m_k \le M_k \le M$$

$$\Rightarrow \sum_{k=1}^{n} m\,\Delta\alpha_k \le \sum_{k=1}^{n} m_k\,\Delta\alpha_k \le \sum_{k=1}^{n} M_k\,\Delta\alpha_k \le \sum_{k=1}^{n} M\,\Delta\alpha_k$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

then,

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le \int_a^b f\,d\alpha \le M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m \le \frac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)} \le M$$

$$\Rightarrow m \le c \le M \quad \text{where } c = \frac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)}$$

Consider $c = \dfrac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)}$

$$\Rightarrow \int_a^b f\,d\alpha = c\{\alpha(b)-\alpha(a)\} = c\int_a^b d\alpha(x)$$

$$\Rightarrow \int_a^b f\,d\alpha = c\int_a^b d\alpha(x) = c\{\alpha(b)-\alpha(a)\}$$

If f is continuous on $[a,b]$ then the range of f is $[m,M]$

Since $m \le c \le M$, by intermediate value theorem, there exist a point $x \in [a,b]$ such that $f(x_0) = c$